A brief introduction to the Lorenz '63 system

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1 Introduction

Next to relativity, quantum mechanics and cosmology, chaos is probably one of the most groundbreaking discoveries in 20th-century theoretical physics. Although some foundational work had already been done much earlier by mathematicians such as Poincaré, a big breakthrough was Edward Lorenz' 1963 paper on convection [4]. We will go through the derivation that lead Lorenz to the equations that are named after him and look at some basic properties of chaotic systems.

Previously we have studied the Rayleigh-Bénard model of convection in a fluid constrained by two horizontal plates at different temperatures. Currently, the full solution of this problem is much too hard to be tackled analytically[1]. Instead we need to resort to simplifications to gather some insight on the behaviour of the equations. We have seen through linear stability analysis that the zero-velocity solution with linear vertical temperature gradient remains stable as long as the parameters of the system are such that the Rayleigh number is small enough. When the Rayleigh number exceeds a critical threshold, we know that the system becomes unstable to perturbations of certain frequencies. This analysis does not tell us however how perturbations develop beyond the linear regime. The full set of equations is non-linear, so after a short time the non-linear terms that have been dropped in the linear analysis will start to contribute to the development of the perturbation.

To study the effects of these non-linearities, Lorenz considered a simplified setting [4], that allows for a more in-depth mathematical and numerical analysis. First of all he takes the flow to be 2-dimensional by taking the velocity in one of the horizontal directions to be zero. Furthermore, in the other horizontal direction the flow is considered to be periodic. This consideration, together with the fact that the vertical motion is restricted by the two plates, allows for a double Fourier expansion of the temperature and velocity fields. By numerically studying the evolution of a large number of Fourier components, Saltzman [5] found evidence that when the Rayleigh number is slightly supercritical, only three of the Fourier modes would remain significantly different from zero. For this reason Lorenz suggested to study a dynamical system containing only these

[1] The Navier-Stokes equations are in fact so hard to analyse that even proving the existence and smoothness of a solution can earn you a US$1,000,000 prize.
http://www.claymath.org/millennium/

three modes. The derivation of this system of equations will be discussed in Section 3. In general, this method of reducing a continuous dynamical system by truncating the number of components in a suitably chosen basis is called the Galerkin method.

Even though the severe Fourier truncation that Lorenz performed is not strictly mathematically motivated, the resulting dynamical system does exhibit some captivating features that strongly resemble properties of the actual atmospheric motion. The most striking property is what is referred to as chaoticity. This term refers to the exponential growth of small errors, making weather forecasts extremely difficult as the time into the future increases. While Lorenz' analysis is mathematically quite straightforward, his discovery has sparked a large interest from the mathematics community since [3]. In Section 2 we will look at some of the signatures of the chaotic behaviour in the Lorenz system.

As the original article by Lorenz is an example of clear scientific writing, I will gratefully make use of much of the reasoning found in the article, with some added explanation. A well written introduction to chaos in geophysics that does not require knowledge of advanced mathematics can be found in [2].

2 Derivation of the Lorenz equations

As equations of motion for the fluid we use the Bousinesq approximation, which are given by

$$\frac{\partial}{\partial t}\underline{u} + (\underset{\sim}{u}.\underset{\sim}{\nabla})\underline{u} = \nu\Delta\underline{u} - \frac{\nabla p}{\rho_0} + (\alpha\theta)\underline{g} \tag{1}$$

$$\frac{\partial}{\partial t}\theta - u_z\Gamma + \underline{u}.\underline{\nabla}\theta = \kappa\Delta\theta \tag{2}$$

$$\underline{\nabla}.\underline{u} = 0 \tag{3}$$

where $\underline{u} = (u_x, u_y, u_z)$ is the velocity field, $\underline{g} = (0, 0, -g)$ is the gravitational acceleration, θ is the departure of the temperature from the linear temperature profile $T(z) = T_H - \Gamma z$, $\Gamma = (T_H - T_L)/d$ and $\nabla = (\partial_x, \partial_y, \partial_z)$ is the gradient operator. Repeated underlinings such as $\underline{.}$ or $\underset{\sim}{.}$ mean that the indices are contracted (i.e. $\underline{u}.\underline{\nabla} = u_x\partial_x + u_y\partial_y + u_z\partial_z$). The boundary conditions are chosen as in the Rayleigh-Bénard problem:

$$u(x, y, z) = 0 \text{ for } z = 0, d \tag{4}$$

$$\theta(x, y, z) = 0 \text{ for } z = 0, d \tag{5}$$

In the linear analysis of the Rayleigh-Bénard problem, the advection terms $(\underset{\sim}{u}.\underset{\sim}{\nabla})\underline{u}$ and $\underline{u}.\underline{\nabla}\theta$ could be dropped as they are of second order in the perturbed variables and hence give non-linear contributions. Here we will not drop these terms, resulting in a richer behaviour of the dynamical system, but also a mathematically more challenging set of equations.

The first simplification that we perform is to assume that there is no movement in the y direction. Hence the velocity field $\underline{u}$ is of the form

$$\underline{u} = (u_x, 0, u_z) \tag{6}$$

Using the fact that the velocity field represents an incompressible flow, i.e. $\underline{\nabla}.\underline{u} = 0$, we can conveniently write the velocity field as derivatives of a stream function $\phi(x, z)$:

$$\underline{u} = (-\partial_z \phi, 0, \partial_x \phi) \tag{7}$$

As we have a two dimensional flow eliminating the pressure term is easier then before. The x component of Eq. 1 contains $\partial_x p$, the z component $\partial_z p$. Hence by differentiating the former with respect to z and the latter with respect to x, we have two equations containing a $\partial_x \partial_z p$ term, which drops out on subtraction. Substituting the stream function into this equation results in

$$\partial_t \Delta\phi - \partial_z \phi \partial_x(\Delta\phi) + \partial_x \phi \partial_z(\Delta\phi) = \nu \Delta^2 \phi + g\alpha \partial_x \theta \tag{8}$$

Similarly substituting Eq. 7 for the stream function into the temperature equation, Eq. 2, we get

$$\partial_t \theta - \partial_z \phi \partial_x \theta + \partial_x \phi \partial_z \theta = \kappa \Delta\theta + \Gamma \partial_x \phi \,. \tag{9}$$

Now we make a further simplification. We assume that both the field ϕ and θ are periodic functions with a period of $2l$ in x and therefore can be decomposed in a Fourier series.

$$\theta(x, z) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=1}^{\infty} \theta_{k_1 k_2} e^{i\pi k_1 x/l} \sin(\pi k_2 z/d) \tag{10}$$

$$\phi(x, z) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=1}^{\infty} \phi_{k_1 k_2} e^{i\pi k_1 x/l} \sin(\pi k_2 z/d) \tag{11}$$

with $\theta_{k_1 k_2} = \bar{\theta}_{-k_1 k_2}$ and $\phi_{k_1 k_2} = \bar{\phi}_{-k_1 k_2}$ to ensure that θ and ϕ are real. If we substitute these equations into Eqs. 8 and 9, using the orthogonality of the Fourier basis, an infinite system of differential equations for the coefficients $\theta_{k_1 k_2}$ and $\phi_{k_1 k_2}$ can be derived. From this system of coupled equations, it can be derived that the most severe truncation of Fourier modes to retain nonlinearity is to retain θ_{11} and θ_{02} as real coefficients and ϕ_{11} as a purely imaginary coefficient. With a rescaling to obtain nondimensional equations, we have

$$\phi(x, z, t) = \frac{(1-a^2)\kappa}{a} \sqrt{2} X(t) \sin(\pi x/l) \sin(\pi z/d) \tag{12}$$

$$\theta(x, z, t) = \frac{R_c (T_H - T_L)}{\pi R} \left(\sqrt{2} Y(t) \cos(\pi x/l) \sin(\pi z/d) - Z(t) \sin(2\pi z/d) \right) \tag{13}$$

where $a = d/l$, $R = \frac{g\alpha d^3 (T_H - T_L)}{\nu\kappa}$ and $R_c = \pi^4 (1 + a^2)^3 / a^2$. By writing out the differential equations for X, Y and Z, using the orthogonality of sine and cosine functions and rescaling the time by $\tau = \pi^2 \frac{(1+a^2)}{d^2} \kappa t$, we finally get the Lorenz '63 equations:

$$\dot{X} = -\sigma X + \sigma Y \tag{14}$$

$$\dot{Y} = -XZ + rX - Y \tag{15}$$

$$\dot{Z} = XY - bZ \tag{16}$$

with time derivatives in τ, $r = R/R_c$ the rescaled Rayleigh number, $b = 4/(1 + a^2)$ and $\sigma = \nu/\kappa$ the Prandtl number. Note that if one more mode (X, Y or Z) would be removed, the system would no longer be non-linear.

3 Analysis of the dynamical system

From the analysis of fixed points of the Lorenz equations and their stability, we can already understand a bit about the Lorenz system, without having to resort to numerical simulations. If we write Eq. 16 in the form of $d(X,Y,Z)/dt = F(X,Y,Z)$ then the fixed points are determined by finding the solutions to $F(X,Y,Z) = 0$. It is easy to calculate that these points are $r_0 = (0,0,0)$ and $r_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. As we are working with real coordinates X, Y and Z, the latter two points are only valid fixed points for $r \geq 1$.

Now we can study the stability of these fixed points. This can be accomplished by looking at the Jacobian of field F in the point in question (see App. A). The Jacobian is given by

$$J(X,Y,Z) = \begin{pmatrix} \partial_X F_X & \partial_Y F_X & \partial_Z F_X \\ \partial_X F_Y & \partial_Y F_Y & \partial_Z F_Y \\ \partial_X F_Z & \partial_Y F_Z & \partial_Z F_Z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ -Z+r & -1 & -X \\ Y & X & -b \end{pmatrix} \tag{17}$$

By calculating the eigenvalues of the Jacobian we can determine the effect of infinitesimal perturbations around the fixed point. If the real parts of all eigenvalues are negative, the fixed point is stable. An analysis of the eigenvalues of $J(r_0)$ shows that the eigenvalues are real whenever $r > 0$ and one eigenvalue turns positive when $r > 1$. Correspondingly, calculating the eigenvalues of $J(r_\pm)$ shows that when $r > 1$, there is one real eigenvalue and two complex conjugate eigenvalues. As long as $\sigma > b+1$ there exists a critical value for r of

$$\sigma \frac{\sigma + b + 3}{\sigma - b - 1}$$

below which the real parts of the eigenvalues are all negative and above which two conjugate complex eigenvalues both acquire a positive real part.

How can we interpret the results we have found up to now? As long as $0 < r < 1$ the only stable fixed point is the origin. This is in agreement with the linear stability analysis of Rayleigh-Bénard convection that showed that as long as $R < R_c$ there are no growing perturbations. When r exceeds 1 the origin looses its stability. At the same time two stable fixed points appear, namely $r_\pm$. So for a slightly supercritical Rayleigh number, steady convection rolls develop, the sense of the flow is dependent upon the initial conditions. What happens however when these two new fixed points loose their stability? The eigenvalues of $r_\pm$ are now one real negative and two conjugate complex eigenvalues with positive real part. In the direction of the eigenvector corresponding to the purely real eigenvalue points are attracted towards $r_\pm$. In the plane corresponding to the eigenvectors with conjugate complex eigenvalues, points make a spiralling motion, moving away from $r_\pm$.

To find out what happens beyond the linear behaviour described above is quite hard. We can however demonstrate that the movement of the point (X,Y,Z) that gives the configuration of the 2D flow is restrained to a finite volume in the space of variables. This can be seen by performing a change of variables

$$X' = X \tag{18}$$

$$Y' = Y \tag{19}$$

$$Z' = Z - r - \sigma \tag{20}$$

Through this change, the differential equation takes the form

$$dX_i/dt = \sum_{j,k} a_{i,j,k} X_j X_k - \sum_j b_{i,j} X_j + c_i \ , \text{ with } i \in \{1, \dots, M\} \tag{21}$$

where $\sum_{i,j,k} a_{i,j,k} X_i X_j X_k$ vanishes identically, $\sum_{i,j} b_{i,j} X_i X_j$ is positive definite and c_i are constants. Now take Q to be

$$Q = \frac{1}{2} \sum_i X_i^2 \ .$$

If $e_1, \dots, e_M$ are the roots of the equations

$$\sum_i (b_{i,j} + b_{j,i}) e_j = c_i$$

then it is easy to see that

$$\dot{Q} = \sum_i X_i \dot{X}_i = \sum_{i,j} b_{i,j} e_i e_j - \sum_{i,j} b_{i,j} (X_i - e_i)(X_j - e_j)$$

The points that fulfil $\dot{Q} = 0$ lie on an ellipse E. The points that lie inside of the ellipse have $\dot{Q} > 0$. The surfaces of constant Q are concentric sphere around the origin. Consider a sphere S that contains the entire ellipse E. Points that start from the outside of the sphere can only move towards the sphere as they must experience a decay in Q. Points that are inside of S can experience an increase of Q while they move inside of the ellipse E, but they can not escape from the sphere S as that would mean that Q would increase along this trajectory, while on and outside S, Q can only decrease.

A further hint is given by the divergence of the vector field F. This divergence has a constant negative value of $-(\sigma + b + 1)$. The divergence gives the instantaneous contraction or expansion rate of small volumes evolving in the configuration space under the flow described by F (see App. B). As the divergence is everywhere negative, the evolution of a small volume in the (X, Y, Z) configuration space will constantly shrink under the flow of F.

So what happens to the points spiralling away from $r_\pm$? They cannot move infinitely far way from $r_\pm$, as they have to stay inside of the sphere S. As the divergence of the field F is everywhere negative, a naive guess might be that the trajectories move onto a 2D surface, a 1D periodic orbit or a 0D fixed point. The Lorenz system however showed that there is another possibility, namely that the trajectories move on an countably infinite number of 2D surfaces. Computer simulations show that as the points spiral away from for example r_+, they escape from the neighbourhood of r_+ and start moving around r_- instead. There the point will revolve a certain number of times around r_- until it moves far enough away from r_- to fall into the spiralling motion around r_+ again. If you imagine a subset of the configuration space moving under the Lorenz flow, at each revolution around the fixed point a part of the set splits of and get attracted by the plane through the other fixed point. After another revolution around their respective fixed points, both parts of the set get split up again and contracted, continuing to be stretched and folded until they form an infinite layer of planes.

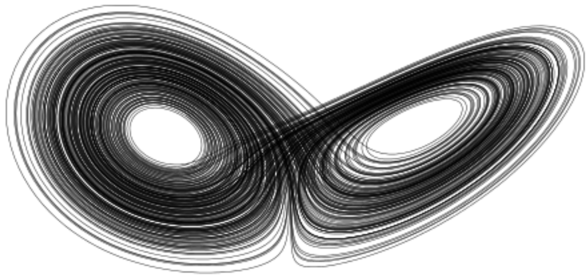

Figure 1: The Lorenz attractor. (From [2])

3.1 Chaos

The anecdote surrounding Lorenz' discovery of chaos is by now part of science folklore: "At one point I wanted to examine a solution in greater detail, so I stopped the computer and typed in the twelve numbers from a row that the computer had printed earlier. I started the computer again, and went out for a cup of coffee. When I returned about an hour later, after the computer had generated about two months of data, I found that the new solution did not agree with the original one. At first I suspected trouble with the computer, which occurred fairly often, but, when I compared the new solution step by step with the older one, I found that at first the solutions were the same, and then they would differ by one unit in the last decimal place, and then the differences would become larger and larger, doubling in magnitude in about four simulated days, until, after sixty days, the solutions were unrecognizably different." [1]

Lorenz soon realized what had happened. The data on the printout which he was copying back into his program had been rounded off slightly. By unknowingly introducing this small error, Lorenz discovered that the system of equations he was studying showed a sensitive dependence on the initial conditions.

To understand a bit more of what is going on, it is more convenient to look at a discretized version of the dynamical system. This will be easier to work with than the full system. Lorenz notes: "The trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit"

The distance from the center, Lorenz proposes to measure using the local maxima of the Z variable. By plotting the next maximum M_{n+1} in function of the previous maximum M_N, he obtains a map that has the shape of a tent. By

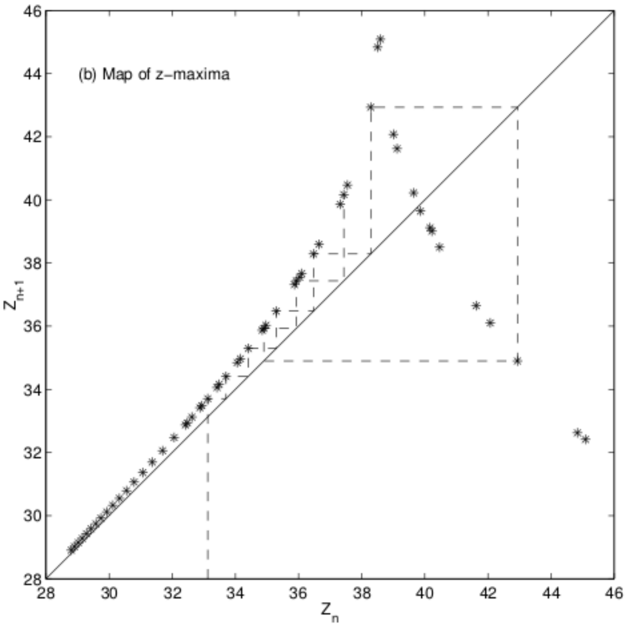

Figure 2: The map of subsequent maxima. (From [2])

making an abstraction of this map, we can calculate some of its properties:

$$M_{n+1} = \begin{cases} 2M_n & \text{for } M_n \leq 1/2 \\ 2(1 - M_n) & \text{for } M_n > 1/2 \end{cases} \tag{22}$$

The sensitive dependence on initial conditions becomes clear in this simple system. If we write the maximum M_n in binary representation

$$M_n = 0.a_1 a_2 a_3 \ldots$$

then the action of the tent map is

$$M_{n+1} = \begin{cases} 0.a_2 a_3 a_4 \ldots & \text{if } a_1 = 0 \\ 0.\bar{a}_2 \bar{a}_3 \ldots \bar{a}_4 & \text{if } a_1 = 1 \end{cases} \tag{23}$$

If an error is introduced by changing the k-th bit a_k to $\bar{a}_k$, this error will shift to a more and more significant position at every iteration. A small error is hence amplified exponentially fast, an error ϵ of 1.2^{-k} is amplified to 1.2^{-k+l} in l iterations. Such exponential divergence is typical in chaotic systems. We see that the exponential rate of growth is here given by $\ln 2$. This growth rate is an example of what is called the Lyapunov exponents of a dynamical system, describing how fast error grow. In this one-dimensional example there is only one exponent. In multi-dimensional systems, more than one exponent will appear and a characteristic of chaotic systems is that the must be one exponent that is positive, indicating the possibility of growing errors.

A Linear stability analysis

Given a dynamical system

$$\dot{x} = F(x) \tag{24}$$

where x is a vector in $\mathbb{R}_n$ of variables and $F : \mathbb{R}_n \to \mathbb{R}_n$ is a vector function. Assume x_0 is a fixed point, i.e. $F(x_0) = 0$. The evolution of perturbations δx around x_0 are given by a series expansion:

$$\frac{d}{dt}(x_0 + \delta x) = F(x_0 + \delta x) = F(x_0) + J(x_0).\delta x + O(\delta x^2) \tag{25}$$

where

$$J(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \dots & \frac{\partial F_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(x) & \dots & \frac{\partial F_n}{\partial x_n}(x) \end{pmatrix} \tag{26}$$

is the Jacobian matrix of the function F. The eigenvalues of J determine the evolution of perturbations around x_0. The real part of the eigenvalues indicate the growth or decrease of the perturbations, depending on whether they are positive or negative. The imaginary part of the eigenvalue indicates a rotation around the fixed point.

If the dynamical system is non-linear, linear stability analysis can only tell us something about the initial behaviour of perturbations. Consider for example the simple one-dimensional system

$$\dot{x} = \mu x - x^3 \tag{27}$$

The origin is a fixed point of this system. If $\mu > 0$, perturbations grow infinitely in the corresponding linearized system

$$\dot{x} = \mu x \tag{28}$$

In the full system however, the behaviour can be very different. The points $\pm\sqrt{\mu}$ are stable fixed points, across which the perturbations cannot grow. If μ is small, the range in which the linear stability correctly predicts the behaviour is also small.

In a higher dimensional system, more complicated behaviour can emerge. Consider the following system in spherical coordinates:

$$\dot{r} = \mu r - r^3 \tag{29}$$

$$\dot{\theta} = \omega + br^3 \tag{30}$$

If $\mu > 0$, the origin is again an unstable fixed point. In Cartesian coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ the eigenvalues of the Jacobian in the origin are $\mu \pm i\omega$, indicating a spiraling motion around the origin, as the linearized set of equations would indicate:

$$\dot{r} = \mu r \tag{31}$$

$$\dot{\theta} = \omega \tag{32}$$

The full system of equations however does not spiral infinitely out from the origin, as around $r = \sqrt{\mu}$ the non-linear term starts to dominate and the trajectory moves into a periodic motion on a circle with radius $\sqrt{\mu}$.

The creation or annihilation of fixed points, periodic orbits or chaotic attractors (as in the Lorenz system) upon the alteration of a parameter is called a bifurcation. For a readable, more in-depth description of the above and more bifurcations, see Strogatz' book [6].

B Evolution of volumes

Say we look at the time evolution of a subset Ω_0 in the coordinate space of a dynamical system

$$\dot{x} = F(x) \tag{33}$$

The volume V_0 of the subset Ω_0 at time $t = 0$ is given by

$$V_0 = \int_{\Omega_0} dP \tag{34}$$

After a short time t the set Ω_0 transforms under the evolution of F to a set Ω_t. The volume of this set is

$$V_t = \int_{\Omega_t} dP' \tag{35}$$

A point P' in Ω_t must originate at time $t = 0$ in a point P in Ω_0. Hence if t is small, we can expand P^prime around P

$$P' = P + F(P).t + O(t^2) \tag{36}$$

Thus the integral over P' in Eq. 35 can be written as an integral over P where Eq. 36 gives the change of variables. The Jacobian determinant for this change is

$$\det(\mathbb{1} + J.t + O(t^2)) = 1 + \mathrm{Tr}(J)t + O(t^2) \tag{37}$$

So we have

$$V_t = V_0 + t\int_{\Omega_0} \mathrm{Tr}(J(P))dP + O(t^2)) \tag{38}$$

$$= V_0 + t\int_{\Omega_0} \nabla.F(P)dP + O(t^2)) \tag{39}$$

showing that the divergence $\nabla.F$ determines the short time evolution of coordinate volumes under F.

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